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Critical thresholds in hyperbolic relaxation systems

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ABSTRACT

Critical threshold phenomena in one-dimensional 2×2 quasi-linear hyperbolic relaxation systems are investigated. Assuming both the subcharacteristic condition and genuine nonlinearity of the flux, we prove global in time regularity and finite-time singularity formation of solutions simultaneously by showing the critical threshold phenomena associated with the underlying relaxation systems. Our results apply to the well-known isentropic Euler system with damping. Within the same framework it is also shown that the solution of the semi-linear relaxation system remains smooth for all time, provided the subcharacteristic condition is satisfied.

Contents

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1. Introduction

It is generic phenomena that solutions of homogeneous systems of quasi-linear hyperbolic conservation laws break down after a finite elapse of time. A balance is often attained with the presence of various sources $S[U]$,

$$U_t + \nabla_x \cdot F(U) = S[U],$$

in which $F(U)$ is the flux and the source $S[U]$ may prevent finite time breakdown from happening if initial configuration is above certain threshold, see [4] for the notion of critical thresholds.

Our main point of interest in this paper is to investigate the critical threshold phenomena associated with hyperbolic relaxation systems. Such a study enables us to gain a global picture on both global in time regularity and the finite time breakdown within one framework.

To be specific, we consider a relaxation system of the form

$$v_t - u_x = 0, \tag{1.1}$$

$$u_t + p(v)_x = \frac{1}{\tau}\big(u_e(v) - u\big), \tag{1.2}$$

$t > 0$, $x \in \mathbb{R}$, subject to bounded and differentiable initial data

$$(v, u)(x, 0) = (v_0, u_0)(x), \quad x \in \mathbb{R}, \tag{1.3}$$

where v and u are scalars, $u_e(v)$ is the equilibrium flux, $p(v)$ is the pressure, and $\tau > 0$ is a relaxation parameter. This system is an important example of a class of 2×2 hyperbolic relaxation systems.

From now on we impose the following basic assumption.

Assumption 1. For all v under consideration it holds:

- $p'(v) < 0$ and $p''(v) > 0$;
- the subcharacteristic condition

$$\lambda_1(v) \leqslant -u_e'(v) \leqslant \lambda_2(v), \tag{1.4}$$

  where λ_i are eigenvalues of system (1.1), (1.2)

$$\lambda_1 = -\lambda_2 = -\sqrt{-p'(v)}. \tag{1.5}$$

The assumption $p'(v) < 0$ ensures the strict hyperbolicity of the system, and $p''(v) > 0$ implies that both characteristic fields are genuinely nonlinear.

It is known that the subcharacteristic type condition (1.4) is necessary even for linear stability as evidenced by Whitham's work [30]. The subcharacteristic condition for a class of 2×2 relaxation systems is coined in [25] for nonlinear stability of shock waves. There have been studies on stability structure conditions for more general relaxation systems [1,32] as well as relaxation methods for constructing weak solutions for conservation laws [8,9,27]. Particularly there has appeared a remarkable development of the stability theory for various relaxation systems in past decades, see, e.g., [7,12,13, 19–21,25,31,33–35], all relying on certain dissipative stability conditions and smallness restriction on initial perturbations.

The classical stability analysis often gives only the solution behavior near some special wave patterns, however, it does not tell us what large perturbations are allowed before losing stability. Recently there have been some research activities geared towards understanding the fully nonlinear dynamics of the systems via a new notion of critical threshold (CT), which serves to describe the conditional stability for underlying physical problems. It is shown that the CT phenomena does reflect the delicate balance among various forcing mechanisms. These include CTs for Euler–Poisson systems [4,23,29], rotating shallow water equations [2,24], restricted Euler equations [22] as well as nonlocal dissipative and dispersive models [5,16–18], among others. Many of the ideas can be traced back to the work [4] by Engelberg, Liu and Tadmor.

Along this direction in [14,15], we obtained both large time regularity and finite time breakdown for a specific traffic flow model—a 2×2 hyperbolic relaxation system. We took advantage of a special feature of the underlying traffic model, namely, the flux $p(v)$ is related to the equilibrium velocity $u_e(v)$ in a specific way [14,15]. This special feature enabled us to decouple slope dynamics of the Riemann invariants and track the dynamics of the whole system effectively.

In this work we consider a more general 2×2 relaxation system (1.1), (1.2). We are still concerned with both global in time regularity and finite-time singularity in solutions by identifying proper threshold conditions. The thresholds are represented in terms of the initial slopes of the Riemann invariants and the initial density.

To present our main results, we now introduce the following notations:

$$r^{\pm} = \left(-p'(v)\right)^{1/4}\left(u_x \mp \sqrt{-p'(v)}\,v_x\right) \tag{1.6}$$

and choose a fixed v^* such that the following quantities

$$G^{\pm}(v) := \frac{1}{2\tau}\int_{v^*}^{v}\left(1 \mp \frac{u_e'(s)}{\sqrt{-p'(s)}}\right)\left(-p'(s)\right)^{1/4} ds - \frac{2}{\tau p''(v)}\left(1 \pm \frac{u_e'(v)}{\sqrt{-p'(v)}}\right)\left(-p'(v)\right)^{5/4} \tag{1.7}$$

are well defined.

The main result of this paper can be stated as follows.

Theorem 1.1. *Consider the relaxation system* (1.1), (1.2) *subject to C^1 bounded initial data $(v_0, u_0)(x)$. Under Assumption* 1, *there exist $C_1 < C_2$, depending only on initial data (v_0, u_0), such that*

$$C_1(u_0, v_0) \leqslant v(x,t) \leqslant C_2(u_0, v_0), \quad \forall x \in \mathbb{R},$$

for $t \geqslant 0$ as long as the C^1 solution exists. Furthermore:

(i) *If for at least one point $x \in \mathbb{R}$ either*

$$r^{+}(x,0) < -\frac{1}{2\tau}\int_{v^*}^{v_0(x)}\left(1 - \frac{u_e'(s)}{\sqrt{-p'(s)}}\right)\left(-p'(s)\right)^{1/4} ds + \inf_{v\in[C_1,C_2]} G^{+}(v)$$

or

$$r^{-}(x,0) < -\frac{1}{2\tau}\int_{v^*}^{v_0(x)}\left(1 + \frac{u_e'(s)}{\sqrt{-p'(s)}}\right)\left(-p'(s)\right)^{1/4} ds + \inf_{v\in[C_1,C_2]} G^{-}(v)$$

holds, then the solution must develop a finite-time singularity where either r^+ or r^- goes to $-\infty$.

(ii) *In addition to Assumption* 1, *if the amplitude of initial data (u_0, v_0) is such that*

$$\inf_{v\in[C_1,C_2]}\left(\frac{\lambda_2^{3/2}(v)}{p''(v)}\left(\lambda_2(v) \pm u_e'(v)\right)\right) \geqslant \frac{1}{4}\int_{C_1}^{C_2}\left(\lambda_2(s) \mp u_e'(s)\right)\lambda_2^{-1/2}(s)\, ds, \tag{1.8}$$

then the solution remains smooth for all time, provided that for all $x \in \mathbb{R}$ it holds

$$r^{\pm}(x,0) \geqslant -\frac{1}{2\tau}\int_{v^*}^{v_0(x)} \left(1 \mp \frac{u_e'(s)}{\sqrt{-p'(s)}}\right)\big(-p'(s)\big)^{1/4}\,ds + \sup_{v\in[C_1,C_2]} G^{\pm}(v). \tag{1.9}$$

Remark 1.1. (i) The thresholds depend only on the relative size of initial configuration and their slopes. Our result goes beyond the scope of classical stability analysis.

(ii) Note that both upper and lower thresholds stated in the theorem are independent of choice of v^*.

(iii) Under condition (1.8), the lower thresholds on the right-hand side of (1.9) are negative, see Remark 2.1. Thus global smooth solutions exist for some initial data with negative Riemann invariant slopes. Furthermore, the magnitudes of the negative slopes are proportional to $\frac{1}{\tau}$ which are not necessarily small. These are in sharp contrast to the generic breakdown in the homogeneous hyperbolic systems [11].

(iv) The condition (1.8) actually requires that the subcharacteristic condition (1.4) is satisfied strictly, i.e., $\lambda_1(v) < -u_e'(v) < \lambda_2(v)$.

Taking $u_e(v) = 0$ in the relaxation system (1.1), (1.2), we obtain the isentropic Euler system with damping:

$$v_t - u_x = 0, \tag{1.10}$$

$$u_t + p(v)_x = -\frac{1}{\tau}u, \tag{1.11}$$

$t > 0$, $x \in \mathbb{R}$.

This system has been studied previously by many authors, e.g., [3,6,28]. Theorem 1.1 yields the critical threshold conditions for (1.10), (1.11). This and other example applications will be presented in Section 5.

Finally we turn to the semi-linear case [9], $p(v) = -\alpha^2 v$, $\alpha > 0$, in the relaxation system (1.1), (1.2),

$$v_t - u_x = 0, \tag{1.12}$$

$$u_t - \alpha^2 v_x = \frac{1}{\tau}\big(u_e(v) - u\big), \tag{1.13}$$

$t > 0$, $x \in \mathbb{R}$.

The subcharacteristic condition (1.4) thus reduces to

$$-\alpha \leqslant -u_e'(v) \leqslant \alpha, \tag{1.14}$$

for v in question. Convergence to equilibrium of this semi-linear system was studied in [26]. Theorem 1.1 does not apply directly since in the proof genuine nonlinearity is essentially used. But formally we see that in this case $G^{\pm} = -\infty$. The solution may not experience any singularity in a finite elapse of time.

In fact, we prove the following result in Section 4.

Theorem 1.2. *Consider the semi-linear relaxation system* (1.12), (1.13) *subject to C^1 bounded initial data* $(v_0, u_0)(x)$. *Under the subcharacteristic assumption* (1.14), *the Cauchy problem* (1.12), (1.13), (1.3) *has a unique C^1 solution for all time $t > 0$.*

We now conclude this section by outlining the rest of this paper. In Section 2 we show the L^∞ bounds for solutions of system (1.1), (1.2), which are essential for estimating the slope dynamics. We then derive a dynamic system for two nonlinear quantities involving solution derivatives and state the critical threshold results. In Section 3, we establish both lower and upper thresholds for a model ordinary differential equation, which when applied to the derived slope dynamics leads to the claimed threshold result. Global smoothness result for semi-linear system (1.12), (1.13) is presented in Section 4. We then apply Theorem 1.1 to obtain critical threshold results for the isentropic Euler system with damping (1.10), (1.11) and to relaxation system (1.1), (1.2) with some specific choices of the pressure and the equilibrium function in Section 5. Concluding remarks are provided in Section 6.

2. Reformulation of the problem

We introduce Riemann invariants of system (1.1), (1.2)

$$R^{\pm} = u \mp m(v), \quad m(v) := \int_{v^*}^{v} \sqrt{-p'(s)}\, ds, \tag{2.1}$$

where $v^* > 0$ is a fixed number, which map (u, v) to (R^-, R^+), and vice versa by

$$u = \frac{1}{2}\left(R^- + R^+\right), \qquad v = m^{-1}\left(\frac{1}{2}\left(R^- - R^+\right)\right). \tag{2.2}$$

Riemann invariants thus satisfy

$$R_t^- + \lambda_1 R_x^- = \frac{1}{\tau}\left(u_e(v) - u\right), \tag{2.3}$$

$$R_t^+ + \lambda_2 R_x^+ = \frac{1}{\tau}\left(u_e(v) - u\right), \tag{2.4}$$

$t > 0$, $x \in \mathbb{R}$, subject to the corresponding initial data

$$R^{\pm}(x,0) = R_0^{\pm}(x) = u_0(x) \mp m\left(v_0(x)\right), \quad x \in \mathbb{R}. \tag{2.5}$$

Through this reformulated system, the existence of a uniform invariant region for the relaxation system (1.1), (1.2) is ensured by the subcharacteristic condition (1.4), as shown in [10].

Lemma 2.1. *Consider the relaxation system* (1.1), (1.2) *subject to data* (1.3). *Let $v_1 < v_2$, and $R_i^{\pm}(v,u)$ be respectively level curves of $R^{\pm}$ through the equilibrium point $(v_i, u_e(v_i))$, $i = 1, 2$. If the subcharacteristic condition* (1.4) *holds, then for any $v \in [v_1, v_2]$, the equilibrium curve $u = u_e(v)$ lies in the domain enclosed by the level curves $R^{\pm}(v,u) = R_i^{\pm}$.*

Again due to the subcharacteristic condition (1.4), the equilibrium curve $u = u_e(v)$ can be expressed as

$$R^+ = \Phi\left(R^-\right),$$

where Φ is a monotone function determined from

$$\frac{R^+ + R^-}{2} = u_e\left(m^{-1}\left(\frac{R^- - R^+}{2}\right)\right).$$

Theorem 2.1. *Consider the relaxation system* (1.1), (1.2) *subject to data* (1.3). *Under Assumption* 1, *we have*:

(i) *If the initial data* (1.3) *lie on equilibrium curve, i.e.,* $u_0 = u_e(v_0)$, *then maximum principle holds for Riemann invariants, i.e.,*

$$\inf_{x\in\mathbb{R}} R_0^{\pm}(x) \leqslant R^{\pm}(x,t) \leqslant \sup_{x\in\mathbb{R}} R_0^{\pm}(x). \tag{2.6}$$

(ii) *For general initial data* (1.3), *the Riemann invariants are bounded by*

$$\min\Big\{\inf_{x\in\mathbb{R}} R_0^-(x), \Phi^{-1}\Big(\inf_{x\in\mathbb{R}} R_0^+(x)\Big)\Big\} \leqslant R^-(x,t) \leqslant \max\Big\{\sup_{x\in\mathbb{R}} R_0^-(x), \Phi^{-1}\Big(\sup_{x\in\mathbb{R}} R_0^+(x)\Big)\Big\}, \tag{2.7}$$

$$\min\Big\{\inf_{x\in\mathbb{R}} R_0^+(x), \Phi\Big(\inf_{x\in\mathbb{R}} R_0^-(x)\Big)\Big\} \leqslant R^+(x,t) \leqslant \max\Big\{\sup_{x\in\mathbb{R}} R_0^+(x), \Phi\Big(\sup_{x\in\mathbb{R}} R_0^-(x)\Big)\Big\}. \tag{2.8}$$

Proof. (i) From Lemma 2.1 it follows that the region enclosed by the level curves $R^{\pm}(v,u) = R_i^{\pm}$ is an invariant region, leading to the stated maximum principle.

(ii) If the initial data do not lie on the equilibrium curve $u = u_e(v)$, we extend the domain to a larger domain so that

$$R_1^{\pm} = \min\Big\{\inf_{x\in\mathbb{R}} R_0^{\pm}(x), \Phi^{-1}\Big(\inf_{x\in\mathbb{R}} R_0^{\mp}(x)\Big)\Big\}, \qquad R_2^{\pm} = \max\Big\{\sup_{x\in\mathbb{R}} R_0^{\pm}(x), \Phi^{-1}\Big(\sup_{x\in\mathbb{R}} R_0^{\mp}(x)\Big)\Big\}.$$

This is then reduced to case (i). □

Theorem 2.1 gives a precise estimate of the lower and upper bounds for both u and v. The bounds for v,

$$C_1(u_0, v_0) \leqslant v(x,t) \leqslant C_2(u_0, v_0),$$

are needed in expressing the threshold curves in terms of some functions of $v \in I$, where the interval I is determined by the initial data with

$$I = \big[C_1(u_0, v_0), C_2(u_0, v_0)\big], \tag{2.9}$$

where

$$\begin{aligned} C_1(u_0, v_0) = m^{-1}\bigg(\frac{1}{2}\Big(&\min\Big\{\inf_{x\in\mathbb{R}}\big(u_0 + m(v_0)\big)(x), \Phi^{-1}\Big(\inf_{x\in\mathbb{R}}\big(u_0 - m(v_0)\big)(x)\Big)\Big\} \\ &- \max\Big\{\sup_{x\in\mathbb{R}}\big(u_0 - m(v_0)\big)(x), \Phi\Big(\sup_{x\in\mathbb{R}}\big(u_0 + m(v_0)\big)(x)\Big)\Big\}\Big)\bigg), \\ C_2(u_0, v_0) = m^{-1}\bigg(\frac{1}{2}\Big(&\max\Big\{\sup_{x\in\mathbb{R}}\big(u_0 - m(v_0)\big)(x), \Phi\Big(\sup_{x\in\mathbb{R}}\big(u_0 + m(v_0)\big)(x)\Big)\Big\} \\ &- \min\Big\{\inf_{x\in\mathbb{R}}\big(u_0 + m(v_0)\big)(x), \Phi^{-1}\Big(\inf_{x\in\mathbb{R}}\big(u_0 - m(v_0)\big)(x)\Big)\Big\}\Big)\bigg). \end{aligned}$$

We now estimate the derivatives of the solution through

$$r^{\pm} = \big(-p'(v)\big)^{1/4} R_x^{\pm} = \big(-p'(v)\big)^{1/4}\big(u_x \mp \sqrt{-p'(v)}\,v_x\big). \tag{2.10}$$

It is clear that the boundedness of (u_x, v_x) is equivalent to the boundedness of $r^{\pm}$ for $v \in I$.

In order to estimate the quantities $r^{\pm}$, we first derive a dynamic system for them. We define

$$a = \frac{1}{4} p''(v)\big(-p'(v)\big)^{-5/4} \tag{2.11}$$

and

$$w = u_e'(v)/\lambda_2(v). \tag{2.12}$$

Assumption 1 implies that for all $v \in I$

$$0 < \inf_{v\in I} a(v) \leqslant a(v) \leqslant \sup_{v\in I} a(v), \qquad |w| \leqslant 1.$$

We further set

$$b^{\pm} = \frac{1 \pm w}{2\tau}, \qquad g^{\pm} = -\frac{1}{2\tau}\int_{v^*}^{v} (1 \mp w)\big(-p'(s)\big)^{1/4}\,ds, \tag{2.13}$$

which satisfy

$$b^{\pm} \geqslant 0, \qquad g^{\pm} \leqslant 0.$$

Lemma 2.2. *The dynamic systems for $r^{\pm}$, defined in* (2.10)*, are*

$$(\partial_t + \lambda_1\partial_x)(r^- - g^-) + a(r^-)^2 + b^- r^- = 0, \tag{2.14}$$

$$(\partial_t + \lambda_2\partial_x)(r^+ - g^+) + a(r^+)^2 + b^+ r^+ = 0, \tag{2.15}$$

$x \in \mathbb{R}$, $t > 0$.

Proof. Set $s^{\pm} = R_x^{\pm}$, and differentiate (2.3) and (2.4) w.r.t. x, respectively, to have

$$s_t^- + \lambda_1 s_x^- + \lambda_{1,v} v_x s^- = \left(\frac{1}{\tau}\big(u_e(v) - u\big)\right)_x,$$

$$s_t^+ + \lambda_2 s_x^+ + \lambda_{2,v} v_x s^+ = \left(\frac{1}{\tau}\big(u_e(v) - u\big)\right)_x,$$

$t > 0$, $x \in \mathbb{R}$.

Here and in what follows λ_v denotes the partial differentiation of λ in terms of v.

Let

$$h = \frac{1}{2}\ln(\lambda_2) = \frac{1}{2}\ln(-\lambda_1). \tag{2.16}$$

Differentiate v along the first characteristic curve $x_1'(t) = \lambda_1$ to have

$$(\partial_t + \lambda_1\partial_x)v =: v' = \frac{(R^-)' - (R^+)'}{2m_v} = \frac{(\lambda_2 - \lambda_1)R_x^+}{2\lambda_2} = s^+. \tag{2.17}$$

Similarly, we have $(\partial_t + \lambda_2\partial_x)v = s^-$. Using again $m(v) = (R^- - R^+)/2$ we obtain

$$\lambda_{1,v} v_x = \frac{\lambda_{1,v}}{2m_v}(s^- - s^+) = h_v(s^+ - s^-).$$

This and (2.17) lead to

$$\lambda_{1,v} v_x = h_v v' - h_v s^- = h' - h_v s^-.$$

Using (2.12), we have

$$\left(\frac{1}{\tau}\big(u_e(v) - u\big)\right)_x = \frac{-1+w}{2\tau} s^- - \frac{1+w}{2\tau} s^+. \tag{2.18}$$

Substitution of these into the above equation for s^- yields

$$(s^-)' + h' s^- - h_v (s^-)^2 = \frac{-1+w}{2\tau} s^- - \frac{1+w}{2\tau} s^+.$$

From (2.10), we see that $r^- = s^- e^h$. This together with (2.17) gives

$$(r^-)' - h_v e^{-h} (r^-)^2 = -b^- r^- - \frac{1+w}{2\tau} e^h v',$$

where b^- is defined in (2.13).

From the definition of g^- in (2.13) and $e^h = \sqrt{\lambda_2} = (-p'(v))^{1/4}$, we see that the last term is nothing but $(g^-)'$. Also we have

$$-h_v e^{-h} = \frac{1}{4} p''(v)\big(-p'(v)\big)^{-5/4} = a > 0,$$

as defined in (2.11). Thus we derived Eq. (2.14).

Similarly, we derive Eq. (2.15).

Lemma 2.2 is thus proved. □

The decoupled dynamics for $r^\pm$ enables us to identify both upper and lower thresholds for the reformulated system. The thresholds for the ordinary differential equations are given in next section in Lemma 3.1. We now proceed to state the threshold conditions for $r^\pm$ to be derived from applying Lemma 3.1 to system (2.14), (2.15).

Theorem 2.2. (i) *If at least at one point $x \in \mathbb{R}$, either*

$$r^+(x,0) < g^+\big(v_0(x)\big) + \inf_{v \in I}\left(- g^+(v) - \frac{b^+(v)}{a(v)}\right)$$

or

$$r^-(x,0) < g^-\big(v_0(x)\big) + \inf_{v \in I}\left(- g^-(v) - \frac{b^-(v)}{a(v)}\right)$$

holds, the solution of system (2.14), (2.15) *must develop singularity at a finite time.*

(ii) *If*

$$\inf_{v \in I}\left(\frac{b^\pm}{a}\right) \geqslant \sup_{v \in I} g^\pm(v) - \inf_{v \in I} g^\pm(v), \tag{2.19}$$

then the solution of system (2.14), (2.15) *remains smooth for all time, provided for all $x \in \mathbb{R}$*

$$r^{\pm}(x,0) \geqslant g^{\pm}\big(v_0(x)\big) + \sup_{v\in I}\left(-g^{\pm}(v) - \frac{b^{\pm}(v)}{a(v)}\right). \tag{2.20}$$

Our main result stated in Theorem 1.1 is a straightforward consequence of Theorem 2.2.

Remark 2.1. We remark that the right-hand side of (2.20) is negative. Indeed, we derive from (2.13) that

$$\begin{aligned} g^{\pm}\big(v_0(x)\big) + \sup_{v\in I}\left(-g^{\pm}(v) - \frac{b^{\pm}(v)}{a(v)}\right) &\leqslant g^{\pm}\big(v_0(x)\big) + \sup_{v\in I}\big(-g^{\pm}(v)\big) + \sup_{v\in I}\left(-\frac{b^{\pm}(v)}{a(v)}\right) \\ &\leqslant \sup_{v\in I} g^{\pm}(v) - \inf_{v\in I} g^{\pm}(v) - \inf_{v\in I}\frac{b^{\pm}(v)}{a(v)} \\ &\leqslant 0, \end{aligned}$$

where the last inequality holds due to condition (2.19).

3. A key lemma

Along each characteristic field, the two Eqs. (2.14), (2.15) for $r^{\pm}$ are ordinary differential equations of the same form

$$\frac{d}{dt}(r-g) + ar^2 + br = 0, \qquad r(0) = r_0,$$

which can be written as

$$\frac{d}{dt}A + a(t)\big(A - b_1(t)\big)\big(A - b_2(t)\big) = 0, \qquad A(0) = A_0, \tag{3.1}$$

with both $a > 0$ and $b_1 \leqslant b_2$ being uniformly bounded for all time.

Lemma 3.1. *Consider Eq.* (3.1) *for A with $\inf a > 0$, $b_1 \leqslant b_2$ and that a, b_1, b_2 are uniformly bounded. We have*:

(i) *If $A_0 < \min b_1$, then solution to* (3.1) *will experience a finite time blow-up at $0 < t_* \leqslant t^* < +\infty$*

$$\lim_{t\to t_*} A(t) = -\infty,$$

where t^* satisfies

$$\int_0^{t^*} a(s)\,ds = \frac{1}{\min b_2 - \min b_1}\ln\left(1 + \frac{\min b_2 - \min b_1}{\min b_1 - A_0}\right)$$

which equals to $\frac{1}{\min b_2 - A_0}$ if $\min b_2 = \min b_1$.

(ii) *If there exists a constant $\bar{b}$ such that*

$$b_1(t) \leqslant \bar{b} \leqslant b_2(t),$$

then (3.1) *admits a unique global bounded solution satisfying*

$$\bar{b} \leqslant A(t) \leqslant \max\{A_0, \max b_2\},$$

provided $A_0 \geqslant \bar{b}$.

Proof. Set $\tau = \int_0^t a(s)\,ds$, which maps $t \in [0, \infty)$ to $\tau \in [0, \infty)$ with $\infty = \int_0^\infty a(s)\,ds$. Then Eq. (3.1) reduces to

$$\frac{d}{d\tau} A + (A - b_1)(A - b_2) = 0, \qquad A(0) = A_0.$$

It is easy to see that $A \leqslant \max\{A_0, \max < b_2\}$.

(i) In order to prove the blow-up result, we consider the following auxiliary problem

$$\frac{d}{d\tau} A^* + \left(A^* - \min b_1\right)\left(A^* - \min b_2\right) = 0, \qquad A^*(0) = A_0,$$

which has only local solution up to τ^* if $A_0 < \min b_1$, with

$$\tau^* = \frac{1}{\min b_2 - \min b_1} \ln\left(1 + \frac{\min b_2 - \min b_1}{\min b_1 - A_0}\right)$$

which equals to $\frac{1}{\min b_2 - A_0}$ if $\min b_2 = \min b_1$.

Let $B = A - A^*$, then it solves the following equation

$$\frac{d}{d\tau} B + B\left(B + 2A^* - b_1 - b_2\right) + C = 0,$$

where

$$\begin{aligned} C &= \left(A^* - b_1\right)\left(A^* - b_2\right) - \left(A^* - \min b_1\right)\left(A^* - \min b_2\right) \\ &= \left(A^* - \min b_1\right)(\min b_1 + \min b_2 - b_1 - b_2) + (b_1 - \min b_1)(b_2 - \min b_1) \geqslant 0, \end{aligned}$$

where $b_2 \geqslant \min b_1$ has been used.

These together lead to

$$\frac{d}{d\tau} B + B\left(B + 2A^* - b_1 - b_2\right) \leqslant 0, \qquad B(0) = 0.$$

Therefore

$$B(\tau) \leqslant 0$$

for as long as the solution B exists.

It follows that the blow-up time τ_* of A is less than or equal to the blow-up time τ^* of A^* since

$$A(\tau) \leqslant A^*(\tau)$$

for as long as both solutions A and A^* exist.

(ii) We now consider an auxiliary problem

$$\frac{d}{d\tau} \bar{A} + (\bar{A} - b_1)(\bar{A} - \bar{b}) = 0, \qquad \bar{A}(0) = A_0,$$

which has a global bounded solution if $\bar{A}_0 \geqslant \bar{b}$ and

$$\bar{A}(\tau) \geqslant \bar{b} \geqslant b_1, \quad \forall \tau > 0.$$

Indeed, $\bar{A}$ satisfies

$$\frac{d}{d\tau}(\bar{A} - \bar{b})e^{\int_0^\tau (\bar{A}(s) - b_1(s))\,ds} = 0$$

which implies that

$$\bar{A} = \bar{b} + (A_0 - \bar{b})e^{-\int_0^\tau (\bar{A}(s) - b_1(s))\,ds} \geqslant \bar{b} > -\infty$$

provided that $\bar{A}_0 \geqslant \bar{b}$.

Let $B = A - \bar{A}$, then it solves the following equation

$$\frac{d}{d\tau}B + B(B + 2\bar{A} - b_1 - b_2) + C = 0,$$

where

$$C = (\bar{A} - b_1)(\bar{b} - b_2) \leqslant 0.$$

These together lead to

$$\frac{d}{d\tau}B + B(B + 2\bar{A} - b_1 - b_2) \geqslant 0, \qquad B(0) = 0.$$

Therefore

$$B(\tau) \geqslant 0, \quad \forall \tau > 0.$$

Hence if $A_0 \geqslant \bar{b}$ we have

$$A \geqslant \bar{A} \geqslant \bar{b},$$

where $\bar{b}$ serves as a lower threshold. This combined with the upper bound $A \leqslant \max\{A_0, \max b_2\}$ leads to the desired estimate. $\square$

Applying this key lemma to Eqs. (2.14), (2.15) for $r^{\pm}$, where

$$A = r^{\pm} - g^{\pm}, \qquad b_2 = -g^{\pm}, \qquad b_1 = -g^{\pm} - \frac{b^{\pm}}{a},$$

taking $\bar{b} = \max b_1$, the desired threshold conditions stated in Theorem 2.2 follow immediately.

4. Global smoothness in semi-linear relaxation system

Consider a semi-linear system

$$v_t - u_x = 0, \tag{4.1}$$

$$u_t - \alpha^2 v_x = \frac{1}{\tau}\big(u_e(v) - u\big), \tag{4.2}$$

$t > 0$, $x \in \mathbb{R}$, where $\alpha > 0$. Assume the subcharacteristic condition (1.14).

The Riemann invariants are

$$R^{\pm} = u \mp \alpha v, \tag{4.3}$$

which map (u, v) to (R^-, R^+), and vice versa by

$$u = \frac{1}{2}\big(R^- + R^+\big), \qquad v = \frac{1}{2\alpha}\big(R^- - R^+\big). \tag{4.4}$$

The Riemann invariants satisfy

$$R_t^- - \alpha R_x^- = \frac{1}{\tau}\big(u_e(v) - u\big), \tag{4.5}$$

$$R_t^+ + \alpha R_x^+ = \frac{1}{\tau}\big(u_e(v) - u\big), \tag{4.6}$$

$t > 0$, $x \in \mathbb{R}$, subject to the corresponding initial data

$$R^{\pm}(x,0) = R_0^{\pm}(x) = u_0(x) \mp \alpha v_0(x), \quad x \in \mathbb{R}. \tag{4.7}$$

The uniform boundedness of $R^{\pm}$ follows from an invariant region under the subcharacteristic condition.

Set $s^{\pm} = R_x^{\pm}$, and differentiate (4.5) and (4.6) w.r.t. x, respectively, to have

$$s_t^- - \alpha s_x^- = \frac{-\alpha + u_e'(v)}{2\tau\alpha} s^- - \frac{\alpha + u_e'(v)}{2\tau\alpha} s^+,$$

$$s_t^+ + \alpha s_x^+ = \frac{-\alpha + u_e'(v)}{2\tau\alpha} s^- - \frac{\alpha + u_e'(v)}{2\tau\alpha} s^+,$$

$t > 0$, $x \in \mathbb{R}$.

Let

$$b = \frac{\alpha - u_e'(v)}{2\tau\alpha}, \qquad g = \frac{\alpha v + u_e(v)}{2\tau\alpha}.$$

A similar calculation as before gives

$$(\partial_t - \alpha\partial_x) g = \frac{\alpha + u_e'(v)}{2\tau\alpha} s^+.$$

Then the equation for s^- becomes

$$(\partial_t - \alpha\partial_x)(s^- + g) + b s^- = 0.$$

Along the first characteristic field $x_1(t) = x_1(0) - \alpha t$, this is just a linear ODE with bounded b. The solution s^- exists for all time and remains bounded. The same claim applies to s^+ too.

Thus we have proved

Proposition 4.1. *Consider the semi-linear system* (4.5), (4.6) *subject to C^1 bounded initial data $(R_0^-, R_0^+)(x)$. Under the subcharacteristic assumption* (1.14), *there exist $K_1 < K_2$, depending only on initial data (R_0^-, R_0^+) such that*

$$K_1 \leqslant R^{\pm}(x,t) \leqslant K_2, \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Furthermore, the C^1 solution remains smooth for all time $t > 0$.

5. Example applications

We now illustrate the critical thresholds stated in Theorem 1.1 for (1.1), (1.2) with some specific choices of the pressure and the equilibrium function.

Example 1. $p(v) = e^{-v}$, $u_e(v) = 0$. In this case we have

$$r^{\pm} = \left(u_x \mp e^{-v/2} v_x\right) e^{-v/4}$$

and the subcharacteristic condition (1.4) is satisfied.

Substituting the pressure and the equilibrium function into (1.7) and taking $v^* = 0$, we have

$$G^{\pm}(v) = 2\left(1 - 2e^{-v/4}\right)/\tau.$$

Now substituting the pressure and the equilibrium function into (1.8), we derive

$$0 < v_{\max} - v_{\min} \leqslant 4 \ln 2.$$

Here and in what follows $v_{\max}$ and $v_{\min}$ denote the global bounds for $v(x,t)$ as identified by C_2 and C_1 in (2.9), depending only on the amplitude of initial data $(v_0, u_0)(x)$.

Corollary 5.1. *Consider the damping system* (1.10), (1.11) *with $p(v) = e^{-v}$, subject to C^1 bounded initial data* (1.3).

(i) *If for at least one point $x \in \mathbb{R}$ either of*

$$u_{0,x} \mp e^{-v_0/2} v_{0,x} < \frac{2}{\tau}\left(1 - 2e^{\frac{v_0(x) - v_{\min}}{4}}\right)$$

holds, then the solution must develop a finite-time singularity when either u_x or v_x becomes unbounded.

(ii) *Assume that $0 < v_{\max} - v_{\min} \leqslant 4 \ln 2$, the solution remains smooth for all time, provided for all $x \in \mathbb{R}$ it holds*

$$u_{0,x} \mp e^{-v_0/2} v_{0,x} \geqslant \frac{2}{\tau}\left(1 - 2e^{\frac{v_0(x) - v_{\max}}{4}}\right).$$

Example 2. $p(v) = \frac{1}{\gamma} v^{-\gamma}$ and $u_e(v) = \frac{1}{1-\gamma} v^{(\gamma-1)/2}$ where $\gamma > 3$. In this case we have

$$r^{\pm} = \left(u_x \mp v^{-\frac{\gamma+1}{2}} v_x\right) v^{-\frac{\gamma+1}{4}}$$

and the subcharacteristic condition (1.4) is satisfied.

Substituting the pressure and the equilibrium function into (1.7) and taking $v^* = 1$, we have

$$G^+(v) = \frac{4}{\tau}\frac{\gamma-2}{(3-\gamma)(\gamma+1)}v^{\frac{3-\gamma}{4}} - \frac{1}{\tau(3-\gamma)}, \qquad G^-(v) = \frac{4}{\tau}\frac{\gamma}{(3-\gamma)(\gamma+1)}v^{\frac{3-\gamma}{4}} - \frac{3}{\tau(3-\gamma)}.$$

Now substituting the pressure and the equilibrium function into (1.8), we derive

$$1 < \frac{v_{\max}}{v_{\min}} \leqslant \left(\frac{4\gamma}{3(\gamma+1)}\right)^{4/(\gamma-3)},$$

where the right-hand side is greater than one when $\gamma > 3$. In the case $1 \leqslant \gamma \leqslant 3$ a similar but different condition can be derived from the additional assumption (1.8).

Corollary 5.2. *Consider the relaxation system* (1.1), (1.2) *with* $p(v) = \frac{1}{\gamma}v^{-\gamma}$ *and* $u_e(v) = \frac{1}{1-\gamma}v^{(\gamma-1)/2}$ *for* $\gamma > 3$, *subject to* C^1 *bounded initial data* (1.3).

(i) *If for at least one point* $x \in \mathbb{R}$ *either*

$$u_{0,x} - v_0^{-(\gamma+1)/2}v_{0,x} < \frac{v_0(x)}{\tau(\gamma-3)}\left(1 - \frac{4(\gamma-2)}{\gamma+1}\left(\frac{v_{\min}}{v_0(x)}\right)^{(3-\gamma)/4}\right)$$

or

$$u_{0,x} + v_0^{-(\gamma+1)/2}v_{0,x} < \frac{v_0(x)}{\tau(\gamma-3)}\left(1 - \frac{4\gamma}{\gamma+1}\left(\frac{v_{\min}}{v_0(x)}\right)^{(3-\gamma)/4}\right)$$

holds, then the solution must develop a finite-time singularity when either u_x *or* v_x *becomes unbounded.*

(ii) *Assume that*

$$1 < \frac{v_{\max}}{v_{\min}} \leqslant \left(\frac{4\gamma}{3(\gamma+1)}\right)^{4/(\gamma-3)},$$

the solution remains smooth for all time, provided for all $x \in \mathbb{R}$ *it holds*

$$u_{0,x} - v_0^{-(\gamma+1)/2}v_{0,x} \geqslant \frac{v_0(x)}{\tau(\gamma-3)}\left(1 - \frac{4(\gamma-2)}{\gamma+1}\left(\frac{v_{\max}}{v_0(x)}\right)^{(3-\gamma)/4}\right)$$

and

$$u_{0,x} + v_0^{-(\gamma+1)/2}v_{0,x} \geqslant \frac{v_0(x)}{\tau(\gamma-3)}\left(1 - \frac{4\gamma}{\gamma+1}\left(\frac{v_{\max}}{v_0(x)}\right)^{(3-\gamma)/4}\right).$$

6. Concluding remarks

We proved global in time regularity and finite-time singularity formation of solutions simultaneously by showing the critical threshold phenomena for p-system with relaxation. In particular, we identified lower thresholds for finite-time singularity in solutions and upper thresholds for the global existence of the smooth solutions. The thresholds are represented in terms of the initial slopes of the Riemann invariants and the initial density, see (1.6). Our results are distinct from previous global stability results for 2×2 hyperbolic relaxation systems. The identified thresholds depend only on the relative size of initial data and their slopes.

We then discussed applications of Theorem 1.1 to relaxation system (1.1), (1.2) with some specific choices of the pressure and the equilibrium function. In particular, we obtained threshold conditions for the isentropic Euler system with damping (1.10), (1.11). Global smoothness for semi-linear system (1.12), (1.13) is also presented.

We hope that our result is proven helpful for understanding nonlinear dynamics in more general systems of quasi-linear hyperbolic conservation laws with sources, which will be one of our future research subjects.

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